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Review Article

THE INTEGRALS OF POISSON AND SCHWARTZ AND THE TRANSFORMATION OF LAPLACE

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ABSTRACT

An article is devoted to a communication of transformation of Laplace with the decision of the task of Dirichlet. With help of integrals of Poisson and Schwartz we obtain an inversion formula of the transformation of Laplace by only positive values of argument. The expression of the sine-transformation of Fourier through the cosine-transformation is found for the functions of hole type. A special decisions of the task of Dirichlet in half-plane is considered.

INTRODUCTION

An article is devoted to a communication of transformation of Laplace with the decision of the task of Dirichlet (Lavrentiev, Shabat, 1987) in the upper half-plane. In opinion of author, the special decision of the task of Dirichlet from the 2,3 lemmas can find an application in physical problems, such as a describing of the distribution of fluid or plasma in various areas. It should be noted that the relationship between these two fundamental concepts of mathematics previously little-researched (in addition to the original definition of the integral Schwartz (Lavrentiev, Shabat (1987), p.209, the (6) equality). Many of the works were devoted to similar problems in terms of the Laplace transform (A.V.Pavlov (2011, 2013, 2014), Andrey Pavlov V. (2014 a,b,c), B.J.Davis. (2002), Sorokin, (2010), Lavrentiev andShabat (1987). We consider a representation of the solution of the task of Dirichlet in the upper half-plane in the form of Poisson's integral (Lavrentiev, Shabat, (1987), p.209, the (5) equality). The integral of Schwartz describes this view in terms of analytical functions with a known valid part of u (x). Thisthe real part is defined on the real line and coincides with the real function. In this situation the integral of Schwarz on complex axis coincides with the transformation of Laplace of the transformation of Fourier of functions u (x). The proposition 1 is devoted to description of the integral of Schwartz in terms of transformation of Laplace. The results of the proposition 1 and the lemmas 1,2 enable to find exact formulas of the transformation of Laplace in the form

$$u(y)=u(-y)=(1/\text{®})\int_0^{\infty}[L(t)[y/(t^2+y^2)]dt,\, y\in(0,+\infty), u(-x)=u(x),$$

$$L(y)=(1/\text{®})\int_0^{\infty}e^{-yt}dt\int_{-\infty}^{\infty}e^{-ist}u(s)ds,\, y\in[0,\infty),$$

if the function u (p) is regular in the all complex plane (a function of integer type). This fact immediately leads to the formula of Laplace transforms only for positive real values of argument (the remark 2 to theorem 1).

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The formula of this remark repeats the author's results contained in articles (Andrey Pavlov V. (2014 a,b,c), A.V.Pavlov (2013, 2014). In these articles the related formula is proved with help of other methods in a somewhat more general situation (not only for functions of integer type). The methods in this article lead not only to new formulas of inversion (the new inversion formula for the transformation of Laplace by only real positive values), but also make it possible to provide an expression for the cosine and sine transformation of Fourier from functions of integer type as

$$\int_0^\infty \cos tx\, Z(t)dt = i\int_0^\infty \sin itx\, Z(t)dt, x \in (0,+\infty).$$

where the right part is an analytical continuation from $p=s\in(0,\infty)$ to the $p=it, t\in(0,+\infty)$ (the theorem 1). The theorem 1 is the main result of this article.The next result of the article is an explicit expression of sine from cosine of the transformation of Fourier of the functions of integer type (the remark 1 to the theorem 1) in the form

$$\int_0^\infty \sin sx\, dx\int_0^\infty \cos tx\,(iu(is))dt = u(s), u(-s) = -u(s), s \in (-\infty,\infty),$$

where $u(p)$ is regular in the all complex plane.

In addition to the listed results the representation of the integral of Schwartz as a real expression on real axis represents an unconditional interest. The fact pointed out in the remark 3 of this article. These kinds of decisions in terms of the task of Dirichlet are important from the point of view of physical applications, but this article is devoted to the exposition of the mathematical facts only for the task of Dirichlet in the half-plane.

THE INTEGRAL OF SCHWARTZ AND THE TRANSFORMATION OF LAPLACE

We prove all the results of the article with help only well-known methods of the complex analysis.

By definition,

$$L(y) = (1/®)\int_0^\infty e^{-yt}dt\int_{-\infty}^\infty e^{-ist}u(s)ds, y \in [0,\infty).$$

We will use the definition of the integral of Schwartz ([6,p.209,the (6) equality])

$$f(p) = (1/i®)\int_{-\infty}^\infty [u(t)[1/(t-p)]dt,\ Im\, p \geq 0.$$

From the equality in the Poisson integral (Lavrentiev, Shabat (1987), p.209, the (5) equality) as a real part of the integral of Schwartz we get the lemma 1. By definition, D_u is the area of regularity of the function

$$u(p), u(-p) = u(p), p \in D_u, C_+ = \{p : Im\, p \geq 0\}.$$

Proposition

$$L(p) = u(pi), Re\, p \geq 0,$$

and

$$f(ip) = L(p), Re\, p \geq 0,$$

if $C_+ \in D_u$, and

$$\int_{-\infty}^\infty |u(x)|\,dx < \infty, \lim_{x\to\pm\infty} u(x) = 0.$$

Proof

From the integral of Schwartz (Lavrentiev, Shabat (1987), p.209, the (6) equality) we get

$$Re\, f(x) = \lim_{p\to x, Im\, p>0} Re\, f(p) = u(x), x \in (-\infty,\infty), p = x+iy.$$

The only regular in C_+ function with the real part $u(x)$ on real axis is $f(p)=u(p)+ic, Im\, p\geq 0, ic=0, u(\pm\infty)=0$, [6,p.209], and we obtain $f(iy)=u(iy), y\in[0,\infty)$,

where

$$f(iy)=(1/i®)\int_{-\infty}^{\infty}[u(t)[1/(t-iy)]dt=\int_{-\infty}^{\infty}[u(t)[1/(y+it)]dt=L(y), y\in[0,\infty);$$

we changed the limits of integration in the definition of the $L(y)$ integral with help of the condition of proposition (Fihtengoltz 1969) . It is obviously (Lavrentiev, Shabat, 1987), $f(ip)=L(p)$ in the area of regularity of $f(ip)=u(ip)$, $ip\in C_+$. The proposition is proved.

Lemma 1

For

$u(x)=u(-x), x\in(-\infty,\infty),$

the equality

$$L(y)=u(iy)=(1/®)\int_{0}^{\infty}[2u(t)[y/(t^2+y^2)]dt, y\in(0,+\infty),$$

takes place, if $C_+\in D_u$, and, if

$$\int_{-\infty}^{\infty}|u(x)|\,dx<\infty.$$

Proof

From the proposition we get $f(iy)=L(y), y\in(0,+\infty)$, where for $u(x)=u(-x), x\in(-\infty,\infty),$ we use equality

$$L(y)=Re\,(1/®)\int_{-\infty}^{\infty}[u(t)[1/(y+it)]dt=(1/®)\int_{0}^{\infty}[2u(t)[y/(t^2+y^2)]dt, y\in(0,+\infty),$$

(we changed the limits of integration in $L(y)$).

We can use the proposition, where $L(y)=u(yi), y\in(0,+\infty),$ and the lemma 1 is proved (in the lemma 1 $u(yi)=Re\,u(yi), y\in([0,+\infty)$).

We obtain, that for the u (ip) function is the analytical continuation of the $L(p)$ function from the right part of the complex axis to the left part.

Lemma 2

For

$u(x)=-u(-x), x\in(-\infty,\infty),$

the equality

$$L(y)=u(iy)=-i(1/®)\int_{0}^{\infty}[u(t)[2t/(t^2+y^2)]dt, y\in(0,+\infty),$$

takes place, if $C_+\in D_u$, and, if

$$\int_{-\infty}^{\infty}|u(x)|\,dx<\infty.$$

Proof.

For $u(-x)=-u(x)$ from the proposition the integral of Shvarch (Lavrentiev, Shabat (1987), p.209, the (6) equality) is equal to

$$f(iy)=L(y)=-i(1/\circledR)\int_{-\infty}^{\infty}[u(t)[t/(t^2+y^2)]dt,\ y>0,$$

(we changed the limits of integration in $L(y)$).

From the lemma 1 for $pu(p)=u_1(p)$ we get

$$L(y)=-i(1/y)(1/\circledR)=u(x)\int_{-\infty}^{\infty}[u_1(t)y/(t^2+y^2)]dt=-i(1/y)u_1(iy)=u(iy), y\in(0,+\infty).$$

We obtain $L(p)=u(ip), Re\, p>0$, and by proposition $f(ip)=u(ip), Re\, p\geq 0$.

The lemma 2 is proved.

Lemma 3

If $u(x), u(-x)=u(x), x\in(-\infty,\infty),$ is regular in all the complex plane, if

$$\int_{-\infty}^{\infty}|d^2u(t)/dt^2|\,dt<\infty, \int_{-\infty}^{\infty}|u(t)|\,dt<\infty,$$

the function

$$g(s)=\int_{0}^{\infty}\sin sx\,dx\int_{0}^{\infty}\cos tx\,u(t)dt, s\in(0,+\infty), g(-s)=-g(s), s\in(-\infty,\infty),$$

is regular in all the complex plane too.

(From the works (Andrey Pavlov V. (2014 a,b,c)) we can obtain the lemma 3, if only the condition

$$\int_{-\infty}^{\infty}|u(t)|\,dt<\infty,$$

takes place, but the proof is substantially simpler).

Proof

From the equality for the Fourier transform (Kolmogorov, Fomin (1976))

$$2\circledR u(s)=\int_{-\infty}^{\infty}e^{isx}\,dx\int_{-\infty}^{-\infty}e^{-itx}u(t)dt=2\circledR u(s), s\in(-\infty,+\infty),$$

we obtain

$$2\circledR u(s)=\int_{0}^{\infty}e^{-isx}\,dx\int_{-\infty}^{\infty}e^{itx}u(t)dt+\int_{0}^{-\infty}e^{isx}\,dx\int_{-\infty}^{\infty}e^{-itx}u(t)dt=$$

$$\lim_{p\to is, Re\,p>0}\int_{0}^{\infty}e^{-px}\,dx\int_{-\infty}^{\infty}e^{itx}u(t)dt+$$

$$+\lim_{p\to is, Re\,p<0}\int_{0}^{\infty}e^{px}\,dx\int_{-\infty}^{\infty}e^{-itx}u(t)dt=L_+(s)+L_-(s), s\in(-\infty,\infty),$$

where

$$L_+(p)=\int_{0}^{\infty}e^{-px}\,dx\int_{-\infty}^{\infty}e^{itx}u(t)dt, Re\,p>0,$$

$$L_{-}(p)=\int_{0}^{\infty} e^{px}\,dx\int_{-\infty}^{\infty} e^{-itx}u(t)dt, Re\, p<0,$$

and we obtain, that

$$2®u(s)-L_{+}(s)=L_{-}(s), s\in(-\infty,\infty).$$

From the theorem about the analytical continuation across the real axis (Lavrentiev, Shabat (1987)) the theorem is proved with help of the theorem of Morer (Lavrentiev, Shabat (1987), p.60) we can define the functions $L_{-}(p), Re\, p<0,$ as $2®u(p)-L_{+}(p), Re\, p>0,$ and we obtain

$$2®u(p)-L_{+}(p)=L_{-}(p), p\in C,$$

where C is the all complex plane - we use , that the $L_{+}(p), L_{-}(p), u(p)$ functions are continuous on the boundary $(-\infty,\infty)$. The fact is proved (G.M.Fihtengoltz (1969)) with help of the obvious equalities

$$|\int_{-\infty}^{\infty} e^{itx}u(t)dt|=|(1/(ix)^{2})\int_{-\infty}^{\infty} e^{itx}(d^{2}u(t)/dt^{2})dt|\le(1/x^{2})\int_{-\infty}^{\infty}|d^{2}u(t)/dt^{2}|\,dt=c/x^{2}, x\to\pm\infty.;$$

$$\int_{-\infty}^{\infty}|e^{itx}u(t)|\,dt|\le\int_{-\infty}^{\infty}|u(t)|\,dt,$$

$$\int_{-\infty}^{\infty}|u(x)|\,dx<\infty.$$

We have proved, that the $L_{-}(p), L_{+}(p)$ functions are regular in the all complex plane.

The lemma 3 is proved.

Theorem 1

$$iu_{0}(ix)=u(x), x\in(0,+\infty),$$

if

$$u(t)=\int_{0}^{\infty} Z(x)(\cos tx)dt, x\in[0,+\infty),$$

for a function $Z(x)$ in the conditions of the lemma 3 (the u(x) function mast be regular in all the complex plane with some additional conditions),where

$$u_{0}(x)=\int_{0}^{\infty}\sin tx\,Z(t)dt, x\in(0,+\infty).$$

Proof

If $u(-p)=u(p), p\in C_{+}$,we can define

$$Z(x)=\int_{0}^{\infty}\cos tx\,u(t)dt, x\in[0,+\infty),$$

$$u_{0}(s)=\int_{0}^{\infty}\sin tx\,Z(x)dx=\int_{0}^{\infty}\sin sx\,dx\int_{0}^{\infty}\cos tx\,u(t)dt\,, x\in[0,+\infty).$$

(we can use the inversion formula for the transforms of Fourier).

From

$$\int_0^\infty \cos tx\, u(t)dt = (1/x^2)\int_0^\infty \cos tx\,(d^2u(t)/dt^2)dt \le c_1/x^2, x \to= \infty, \sin 0x = 0,$$

we can use the inversion formula for the transforms of Fourier (Kolmogorov, Fomin (1976)), and the equality

$$Z(x) = \int_0^\infty \sin tx\, u_0(t)dt, x \in [0,+\infty),$$

takes place.

From the lemma 3 the both functions $u_0(p), u(p)$ are regular in the all complex plane and $u_0(-x) = -u_0(x), x \in (-\infty,\infty)$, (in the lemma 3 $g(x) = u_0(x), x \in (-\infty,\infty)$. For both function we can use the lemma 2:

$$L(s) = 2\int_0^\infty e^{-sx}dx\int_0^\infty \cos tx u(t)dt = u(is),\ u(-t) = u(t), t \in (-\infty,\infty),$$

$$2\int_0^\infty e^{-sx}dx\int_{-\infty}^\infty \sin tx\, u_0(t)dt = i\int_0^\infty e^{-sx}dx\int_{-\infty}^\infty i\, sin(-tx)u_0(t)dt =$$

$$= i\int_0^\infty e^{-sx}dx\int_{-\infty}^\infty e^{-itx}u(t)dt = iu_0(is), s \in (-\infty,\infty),$$

Remark 1

We have proved too, that

$$\int_0^\infty \sin sx\, dx\int_0^\infty \cos tx\, u(t)dt = u_1(s), iu_1(is) = u(si), s \in (0,\infty),$$

in the conditions of the lemma 3.

Remark 2

From the lemma 1 (with $u(ix) = L(x)$) we obtain a new inversion formula for the Laplace transforms by only positive values (A.V.Pavlov (2013, 2014), Andrey Pavlov V.(2014 a,b,c) - all with help of other methods) :

$$u(y) = u(-y) = (1/®)\int_0^\infty [L(t)[y/(t^2+y^2)]dt, y \in (0,+\infty),$$

if $u(-p) = u(p), p \in C_+$, and, if

$$\int_{-\infty}^\infty |u(x)|dx < \infty.$$

Proof

We can not use $L(y) < const./y^2, y \to \infty,$ but

$$(2/®)\int_0^\infty L(t)[y/(t^2+y^2)]dt = \mathrm{Re}\, f(iy) = f(iy), y \in [0,+\infty),$$

without the change of limits of integration with help of the definition of the integral of Poisson (Lavrentiev , Shabat (1987), p.209, the (5) equality) for $p = iy, \mathrm{Re}\, p > 0, y \in [0,+\infty),$ if

$$u(-iy) = L(-y) = L(y) = u(iy), y \in [0, +\infty), \lim_{y \to \pm\infty} u(iy) = 0,$$

(we can use the condition from the lemma 1, the last condition is obvious). Now, we can use lemma 1 (the only regular in C_+ function with the real part $L(x)$ on the real axis is $f(p), L(\pm\infty) = 0$), and

$$\mathrm{Re}\, f(iy) = f(iy) = L(iy), L(iy) = u(i(iy)) = u(-y), y \in (0, +\infty).$$

The remark 2 is proved.

In the works (A.V.Pavlov (2011, 2013, 2014), Andrey Pavlov V. (2014 a,b,c)) the formula is proved by other methods for the more wide class of functions.

Remark 3

From the lemma 2 (in the condition of the lemma 2) we obtain the interesting fact:

$$Im(f(x)) \equiv 0, x \in (-\infty, \infty),$$

if $Re(f(x)) \equiv u(x), Re(u(x)) = u(x) x \in (-\infty, \infty),$ where $u(-x) = -u(x) x \in (-\infty, \infty),$ but $Im(f(ix)) = u(ix) \neq 0, x \in (-\infty, \infty).$

Conclusion

In conclusion, we note, that the principal role of this theme is the Poisson integral Lavrentiev, Shabat (1987), p.209, the (5) equality), comparison with which immediately leads to a non-trivial results lemme 1.2. Without the application of integral of Poisson the results of 2,3 lemmas impossible to prove by any other methods. However, the use of the task of Dirichlet and the integral of Schwartz (Lavrentiev, Shabat (1987), p.209, the (6) equality) directly without a complicated calculations will have very unexpected results (lemma 2-the view of the imaginary part of the decision). In conclusion, the author would like to point out, that not only in the situation of the article for the task of Dirichlet (Lavrentiev, Shabat (1987)) the similar unforeseeable results appeared (note the work (Pavlov A.V. (2014), Andrey Pavlov V. (2014 c)). It is also important to note, that the results do not seem to be unpredictable by themselves without the use of the concepts of the odd ore even functions (A.V.Pavlov (2013, 2014), Andrey Pavlov V. (2014 a,b,c)).

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